Algebraic inequality with trigometric variations.
https://www.linkedin.com/feed/update/urn:li:activity:6555020905666482176
Let $x,y,z > 0$ such that $x+y+z = 1$. Show that

$$\sum(1-x)\sqrt{3yz(1-y)(1-z)} \geq 4\sqrt{xyz}.$$

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After homogenization inequality of the problem becomes

(1) $\sum(y+z)\sqrt{3yz(z+x)(x+y)} \geq 4(x+y+z)\sqrt{(x+y+z)xyz}$.

Let ABC be a triangle with side lengths $a := y+z, b := z+x, c := x+y$, semiperimeter $s = x+y+z$, area $F = \sqrt{(x+y+z)xyz}$ and circumradius R .

Then (**1**)$\Leftrightarrow \sum a\sqrt{3(s-b)(s-c)bc} \geq 4sF \Leftrightarrow$

$$\sqrt{3}\,abc\sum\sqrt{\frac{(s-b)(s-c)}{bc}} \geq 4sF \Leftrightarrow \sqrt{3}\cdot 4RF\sum\sqrt{\frac{(s-b)(s-c)}{bc}} \geq 4sF \Leftrightarrow$$

$$\sqrt{3}\cdot R\sum\sqrt{\frac{(s-b)(s-c)}{bc}} \geq s \Leftrightarrow \sum\sqrt{\frac{(s-b)(s-c)}{bc}} \geq \frac{s}{R\sqrt{3}}.$$

Since $\sqrt{\frac{(s-b)(s-c)}{bc}} = \sin\frac{A}{2}$ and $\frac{s}{R} = 4\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$ then latter inequality becomes $\sum\sin\frac{A}{2} \geq \frac{4}{\sqrt{3}}\prod\cos\frac{A}{2} \Leftrightarrow \sum\cos\alpha \geq \frac{4}{\sqrt{3}}\prod\sin\alpha$, where $\alpha := \frac{\pi-A}{2}, \beta := \frac{\pi-B}{2}, \gamma := \frac{\pi-C}{2}$.

Since $\alpha,\beta,\gamma > 0$ and $\alpha+\beta+\gamma = \pi$ then α,β,γ can be considered as angle of some triangle with side lengths, semiperimeter, circumradius and inradius we, for convenience, will denote respectively, via a,b,c,s,R and r (don't mix these notations with used above for the original triangle)

Therefore, $\sum\cos\alpha \geq \frac{4}{\sqrt{3}}\prod\sin\alpha \Leftrightarrow$

$$1+\frac{r}{R} \geq \frac{4}{\sqrt{3}}\cdot\frac{abc}{8R^3} \Leftrightarrow 1+\frac{r}{R} \geq \frac{4}{\sqrt{3}}\cdot\frac{4Rrs}{8R^3} \Leftrightarrow$$

$$1+\frac{r}{R} \geq \frac{2}{\sqrt{3}}\cdot\frac{rs}{R^2} \Leftrightarrow \frac{R(R+r)}{r} \geq \frac{2s}{\sqrt{3}}.$$

Since $s \leq \frac{3\sqrt{3}}{2}R \Leftrightarrow \frac{2s}{\sqrt{3}} \leq 3R$ and $2r \leq R$ then

$$\frac{R(R+r)}{r} - \frac{2s}{\sqrt{3}} \geq \frac{R(R+r)}{r} - 3R = \frac{R(R-2r)}{r} \geq 0.$$

Remark.
Thus original algebraic inequality has the following different equivalent geometric-trigonometric interpretations:

$$\frac{4}{\sqrt{3}}\prod\cos\frac{A}{2} \leq \sum\sin\frac{A}{2};$$

$$\frac{4}{\sqrt{3}}\prod\sin A \leq \sum\cos A;$$

$$abc \leq 2\sqrt{3}R^2(R+r);$$

$$s \leq \frac{\sqrt{3}}{2}\cdot\frac{R(R+r)}{r} \Leftrightarrow F \leq \frac{\sqrt{3}}{2}\cdot R(R+r)$$

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